CWI Quarterly Volume 9 (1 & 2) 1996, pp. 85 – 95

Computational Aspects of Systems over Rings – Reachability and Stabilizability

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In this paper, computational methods to test the reachability and stabilizability of a system over a (polynomial) ring are derived. For a system $\Sigma = (A, B)$ both reachability and stabilizability can be restated as right-invertibility conditions on the matrix $(zI - A \mid B)$ over different rings. After the introduction of a polynomial ideal $\mathcal{I}$ related to the system, both properties can be studied simultaneously. We derive methods to compute a Gröbner basis of the ideal $\mathcal{I}$ and also characterize its variety. In this way we obtain algorithms to verify the reachability of a system over a polynomial ring. The corresponding stabilizability tests are mainly derived for the particular application of time-delay systems with point delays.

1. Introduction

Systems over (polynomial) rings can be seen as a rather straightforward generalization of systems over the field of real numbers. The key-idea of this approach is the observation that a linear time-invariant finite-dimensional system over $\mathbb{R}$ in state-space form, described by

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{cases} \tag{1}$$

is completely characterized by the four real matrices A, B, C, and D. Moreover, intrinsic properties of the system can be translated into properties of the four system defining matrices, and also most design techniques may be carried out, using the quadruple (A, B, C, D) only. Apparently, the quadruple (A, B, C, D) determines the (algebraic) structure of system (1) completely, and in this way, an alternative way of studying the system equations (1) is obtained: one may study quadruples of real matrices of appropriate dimensions instead.

From an algebraic point of view, the restriction that A, B, C, and D are *real* matrices is not necessary. One may also consider quadruples of matrices, whose entries are elements of a ring. In this setting, algebraic operations, like matrix addition and multiplication, are still defined. This motivates the following definition of a system over a ring.

DEFINITION 1.1. ([11]) A (free) *linear system* Σ *over a commutative ring* $\mathcal{R}$ is a quadruple of matrices (A, B, C, D), where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$, and $D \in \mathcal{R}^{p\times m}$, for some integers n, m, and p. □

At first sight it seems strange that in Definition 1.1 no dynamics are involved. However, this is precisely the advantage of this rather general framework: it may be specialized to a lot of interesting situations. A straightforward example are discrete-time systems over a commutative ring $\mathcal{R}$. In this case, the input u, state x, and output y are elements of the $\mathcal{R}$-modules $\mathcal{R}^m$, $\mathcal{R}^n$, and $\mathcal{R}^p$, respectively. Also time-delay systems with point delays may be modeled as a system over a polynomial ring.

EXAMPLE 1.2. Consider a time-delay system (in continuous time) with k incommensurable point delays $\tau_1, \ldots, \tau_k$ and define $\sigma_1, \ldots, \sigma_k$ as the corresponding delay operators:

$$\sigma_i x(t) = x(t - \tau_i), \qquad \sigma_i u(t) = u(t - \tau_i), \qquad (i = 1, \ldots, k).$$

A time-delay system with point delays can then be written as

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(\sigma_1, \ldots, \sigma_k)x(t) + B(\sigma_1, \ldots, \sigma_k)u(t), \\ y(t) & = & C(\sigma_1, \ldots, \sigma_k)x(t) + D(\sigma_1, \ldots, \sigma_k)u(t), \end{array} \right. \tag{2}$$

where $A(\sigma_1, \ldots, \sigma_k)$, $B(\sigma_1, \ldots, \sigma_k)$, $C(\sigma_1, \ldots, \sigma_k)$ and $D(\sigma_1, \ldots, \sigma_k)$ are polynomial matrices in the delay operators $\sigma_1, \ldots, \sigma_k$. Substituting the indeterminates $s_1, \ldots, s_k$ for $\sigma_1, \ldots, \sigma_k$, a quadruple of polynomial matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ in the indeterminates $s_1, \ldots, s_k$ is obtained. Together with the k-tuple of time-delays $\tau_1, \ldots, \tau_k$, this quadruple is a complete description of the original system equations. The quadruple $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ itself can be regarded as a system over the polynomial ring $\mathbb{R}[s_1, \ldots, s_k]$.

The previous examples indicate that a large class of systems fits into the algebraic framework of Definition 1.1. The idea behind this approach is to use the abstract Definition 1.1 to develop a (formal) theory for systems over rings. For this, our intuitive notion what a dynamical system is, remains of utmost importance. An other goal is to generalize design methods, known for systems over fields, to the ring case. Since most design techniques are based on matrix calculations on the quadruple (A, B, C, D), it may be possible to carry out the same operations in the ring case. The advantage of this approach is clear: once a problem has been solved in the abstract algebraic framework, this result can be applied to all types of systems, that can be modeled as a system over a ring.

In this paper we focus on the properties of reachability and stabilizability. First the definitions of these concepts are generalized to the case of systems over rings. In general, the conditions for reachability and stabilizability are difficult to check. However, for systems over *polynomial* rings, this problem can be solved using methods from constructive commutative algebra. We show how reachability and stabilizability can be translated into properties of a polynomial ideal, and how Gröbner basis techniques can be used to test these conditions explicitly.

2. Reachability and stabilizability

Let $\mathcal{R}$ be an integral domain, and let $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$. Then the pair[1] $\Sigma = (A, B)$ is called *reachable* if the columns of the matrix $(B \mid AB \mid \cdots \mid A^{n-1}B)$ span the free module $\mathcal{R}^n$. This generalization of the Kalman rank condition to the ring case coincides with the intuitive notion of reachability for discrete-time systems over rings. The condition can also be formulated as a generalized Hautus test (see [10] or [7, p. 19]):

Proposition 2.1. *Let $\Sigma = (A, B)$ be a system over $\mathcal{R}$. Then (A, B) is reachable if and only if*

$$(zI - A \mid B) \text{ is right-invertible over } \mathcal{R}[z]. \tag{3}$$

□

In the algebraic framework, the notion of stability is somewhat more difficult to describe because the concepts of state and convergence cannot be used explicitly. This problem can be solved by the introduction of so-called Hurwitz sets (see also [3], [9]).

Definition 2.2. A *Hurwitz set* $\mathcal{D}$ is a subset of the polynomial ring $\mathcal{R}[z]$, satisfying the following properties:

(i) $\mathcal{D}$ is *multiplicative*, i.e. $1 \in \mathcal{D}$, and if $p, q \in \mathcal{D}$, then $p \cdot q \in \mathcal{D}$.

(ii) Each polynomial $p \in \mathcal{D}$ is *monic*, i.e. its leading coefficient is equal to 1.

(iii) $\mathcal{D}$ is *saturated*, i.e. if $p \in \mathcal{D}$, and q is monic and divides p, then $q \in \mathcal{D}$.

[1] For the properties of reachability and stabilizability of a system, only the matrices A and B are involved; we may assume that $C = I$ and $D = 0$. Therefore, in the rest of this paper, the matrices C and D are omitted.

(iv) There exists an $\alpha \in \mathcal{R}$ such that $(z - \alpha) \in \mathcal{D}$.

$\square$

One can think of a Hurwitz set as the set of all stable polynomials. For example, for linear time-invariant finite-dimensional continuous-time systems, the set

$$\mathcal{D} = \left\{ p(z) \in \mathbb{R}[z] \mid \forall \lambda \in \overline{\mathbb{C}^+} : p(\lambda) \neq 0 \right\}$$

is the Hurwitz set describing the classical notion of stability. Accordingly, a system $\Sigma = (A, B, C, D)$ over the ring $\mathcal{R}$ is called *internally stable* with respect to a Hurwitz set $\mathcal{D}$, if the characteristic polynomial of the system is a stable polynomial, i.e. if $\det(zI - A) \in \mathcal{D}$.

With a Hurwitz set $\mathcal{D}$ we may associate a ring of fractions, denoted by $\mathcal{R}_{\mathcal{D}}(z)$:

$$\mathcal{R}_{\mathcal{D}}(z) := \left\{ \frac{p(z)}{q(z)} \in \mathcal{R}(z) \mid p(z) \in \mathcal{R}[z] \text{ and } q(z) \in \mathcal{D} \right\}. \tag{4}$$

$\mathcal{R}_{\mathcal{D}}(z)$ is considered as the set of all (not necessarily proper) stable transfer functions. With this ring $\mathcal{R}_{\mathcal{D}}(z)$ in mind, it is possible to generalize the Hautus test for stabilizability to the ring case. However, in this situation *dynamic* state feedback is required to achieve stability. This means that a system $\Gamma = (F, G, H, J)$ over the ring $\mathcal{R}$, i.e. a system of the same type as Σ, is used as a compensator to stabilize Σ.

PROPOSITION 2.3. ([4], [10]) *Let $\Sigma = (A, B)$ be a system over $\mathcal{R}$. Then (A, B) is internally stabilizable with respect to the Hurwitz set $\mathcal{D}$ using dynamic state feedback if and only if*

$$(zI - A \mid B) \textit{ is right-invertible over } \mathcal{R}_{\mathcal{D}}(z). \tag{5}$$

$\square$

Comparing (3) and (5), we see that both reachability and stabilizability are characterized by a right-invertibility condition on the matrix $(zI - A \mid B)$, but over different rings. This motivates a unified approach to study the reachability and stabilizability of a system simultaneously.

3. Translation to polynomial ideals

Right-invertibility properties of the matrix $(zI - A \mid B)$ can be reformulated in terms of a polynomial ideal related to the system $\Sigma = (A, B)$.

DEFINITION 3.1. Let $\Sigma = (A, B)$ be a system over an integral domain $\mathcal{R}$. Then the ideal $\mathcal{I}$ in $\mathcal{R}[z]$ associated with Σ is defined as

$$\mathcal{I} \quad := \quad \{\varphi(z) \in \mathcal{R}[z] \mid \exists M(z) \in \mathcal{R}[z]^{(n+m)\times n} \text{ such that} \tag{6}$$

$(zI - A \mid B) \cdot M(z) = \varphi(z) \cdot I\}.$

$\square$

To some extent, $\mathcal{I}$ can be seen as the ideal describing all internal dynamics that can be obtained from the system $\Sigma = (A, B)$ by dynamic state feedback: if $\varphi(z)$ is a monic polynomial in $\mathcal{I}$, then there exists a dynamic state feedback such that $(\varphi(z))^n$ is the characteristic polynomial of the closed-loop system. The reachability and stabilizability conditions for a system Σ are easily restated as conditions on the associated ideal $\mathcal{I}$.

THEOREM 3.2. *Let $\Sigma = (A, B)$ be a system over an integral domain $\mathcal{R}$, and let $\mathcal{I}$ be the ideal associated with Σ as defined in (6). Then*

(i) $\Sigma = (A, B)$ *is reachable* $\Longleftrightarrow \mathcal{I} = \mathcal{R}[z]$.

(ii) $\Sigma = (A, B)$ *is stabilizable with respect to the Hurwitz set* $\mathcal{D} \Longleftrightarrow \mathcal{I} \cap \mathcal{D} \neq \emptyset$.

$\square$

To test the reachability and stabilizability of a system, we have to find a characterization of the ideal $\mathcal{I}$. For systems over polynomial rings, this can be done using Gröbner basis techniques (see e.g. [2]). Let $\mathcal{K}$ be a field of characteristic zero. In the rest of this paper we assume that $\mathcal{R}$ is a polynomial ring over $\mathcal{K}$ in k indeterminates: $\mathcal{R} = \mathcal{K}[s_1, \ldots, s_k]$.

In its present form, the definition of the ideal $\mathcal{I}$ is not very suitable for Gröbner basis computations. Therefore we introduce two other ideals, strongly related to $\mathcal{I}$, that facilitate the application of the Gröbner basis algorithm.

DEFINITION 3.3. Let $\Sigma = (A, B)$ be a system over $\mathcal{R}$, and denote by e_i the i^{th} unit vector in $\mathcal{R}^n$ $(i = 1, \ldots, n)$. Then the ideals $\mathcal{H}_i$ and $\mathcal{H}$ in $\mathcal{R}[z]$ associated with Σ are defined as

$$\begin{aligned} \mathcal{H}_i \quad := \quad & \{\varphi(z) \in \mathcal{R}[z] \mid \exists \psi(z) \in \mathcal{R}[z]^{n+m} \text{ such that} \\ & (zI - A \mid B) \cdot \psi(z) = \varphi(z) \cdot e_i\}, \end{aligned} \tag{7}$$

$$\mathcal{H} \quad := \quad \bigcap_{i=1}^{n} \mathcal{H}_i. \tag{8}$$

$\square$

The ideal $\mathcal{H}$ can be considered as an alternative column-wise definition of $\mathcal{I}$; it is easily seen that $\mathcal{H} = \mathcal{I}$.

DEFINITION 3.4. Let $\Sigma = (A, B)$ be a system over $\mathcal{R}$, and let $r_1(z), \ldots, r_N(z)$ denote all $n \times n$ minors of the matrix $(zI - A \mid B)$. Then the ideal $\mathcal{J}$ associated with Σ is defined as the ideal in $\mathcal{R}[z]$ generated by all these $n \times n$ minors:

$$\mathcal{J} := \langle r_1(z), \ldots, r_N(z) \rangle_{\mathcal{R}[z]}. \tag{9}$$

□

The next lemma describes the relationship between the ideals $\mathcal{I}$ and $\mathcal{J}$, and their varieties.

LEMMA 3.5. *Let $\Sigma = (A, B)$ be a system over $\mathcal{R}$, and let $\mathcal{I}$ and $\mathcal{J}$ be the ideals associated with Σ as defined in (6) and (9), respectively. Then*

(i) $\mathcal{J} \subset \mathcal{I} \subset \mathrm{rad}(\mathcal{J})$ *(where* $\mathrm{rad}(\mathcal{J})$ *denotes the radical of* $\mathcal{J}$*),*

(ii) $\mathcal{V}(\mathcal{I}) = \mathcal{V}(\mathcal{J})$.

PROOF
(i) "$\mathcal{J} \subset \mathcal{I}$" Let $r(z)$ be one of the $n \times n$ minors of $(zI - A \mid B)$. Then there exists an $n \times n$ sub-matrix $K(z)$ of $(zI - A \mid B)$ such that $r(z) = \det(K(z))$, and according to Cramer's rule we have

$$K(z) \cdot \mathrm{adj}(K(z)) = \det(K(z)) \cdot I = r(z) \cdot I.$$

Extending the matrix $\mathrm{adj}(K(z))$ with zero rows on the right places, we obtain an $(n+m) \times n$ matrix $\tilde{K}(z)$ over $\mathcal{R}[z]$ such that $(zI - A \mid B) \cdot \tilde{K}(z) = r(z) \cdot I$. Hence $r(z) \in \mathcal{I}$. Since $r(z)$ was an arbitrary $n \times n$ minor of $(zI - A \mid B)$, it follows that all principal minors of $(zI - A \mid B)$ belong to $\mathcal{I}$, so $\mathcal{J} \subset \mathcal{I}$.

"$\mathcal{I} \subset \mathrm{rad}(\mathcal{J})$" Let $\varphi(z) \in \mathcal{I}$. Then there exists a matrix $M(z) \in \mathcal{R}[z]^{(n+m)\times n}$ such that

$$(zI - A \mid B) \cdot M(z) = \varphi(z) \cdot I. \tag{10}$$

Let $r_1(z), \ldots, r_N(z)$ denote all $n \times n$ minors of the matrix $(zI - A \mid B)$. Taking determinants on both right- and left-hand side of (10), and using the Binet-Cauchy formula, we find polynomials $\beta_1(z), \ldots, \beta_N(z) \in \mathcal{R}[z]$ (the $n \times n$ minors of the matrix $M(z)$) such that

$$\sum_{i=1}^{N} r_i(z)\beta_i(z) = (\varphi(z))^n.$$

We conclude that $(\varphi(z))^n \in \mathcal{J}$, and thus by definition $\varphi(z) \in \mathrm{rad}(\mathcal{J})$.

(ii) Since $\mathcal{V}(\mathcal{J}) = \mathcal{V}(\mathrm{rad}(\mathcal{J}))$, *(ii)* follows directly from *(i)*.

□

Summarizing, we have the following relationships between the ideals $\mathcal{I}$, $\mathcal{H}$, and $\mathcal{J}$, and their varieties:

(i) $\mathcal{J} \subset \mathcal{I} = \mathcal{H}$,

(ii) $\mathcal{V}(\mathcal{J}) = \mathcal{V}(\mathcal{I}) = \mathcal{V}(\mathcal{H})$.

Although the ideal $\mathcal{J}$ is not exactly the same as $\mathcal{I}$, it has an important advantage: $\mathcal{J}$ is easily characterized using Gröbner basis methods. One just has to apply Buchberger's algorithm (see e.g. [1], [2]) to the minors $r_1(z), \ldots, r_N(z)$ to obtain a Gröbner basis of $\mathcal{J}$. For the ideals $\mathcal{H}_i$ $(i = 1, \ldots, n)$, Gröbner basis computations are more involved, but still possible due to the following lemma:

LEMMA 3.6. ([7, p. 171]) *Let $\Sigma = (A, B)$ be a system over $\mathcal{R}$. Introduce an n-rowvector $q^T = (q_1 \cdots q_n)$ of new indeterminates and define*

$$(p_1 \cdots p_{n+m}) := (q_1 \cdots q_n) \cdot (zI - A \mid B).$$

Consider $p_1, \ldots, p_{n+m}$ as polynomials in $\mathcal{R}[z, q_1, \ldots, q_n]$ and define for $i = 1, \ldots, n$ the ideals $\mathcal{P}_i := \langle p_1, \ldots, p_{n+m} \rangle \cap \mathcal{R}[z, q_i]$. Then

$$\forall i \in \{1, \ldots, n\} : \qquad \mathcal{H}_i = \{\varphi(z) \in \mathcal{R}[z] \mid q_i \cdot \varphi(z) \in \mathcal{P}_i\}. \tag{11}$$

□

Using formula (11), a Gröbner basis for $\mathcal{H}_i$ can be obtained. First a Gröbner basis of $\mathcal{P}_i$ has to be computed, using a specific term ordering that enables one to eliminate the additional indeterminates $q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n$. Then the ideal $\mathcal{H}_i$ is generated by the polynomials $\varphi(z) \in \mathcal{R}[z]$, for which $q_i \cdot \varphi(z)$ belongs to the Gröbner basis of $\mathcal{P}_i$. Finally, the computation of the ideal $\mathcal{H}$ can be carried out by intersecting all ideals $\mathcal{H}_i$ $(i = 1, \ldots, n)$. This is a standard procedure in the theory of Gröbner bases (see e.g. [2, p. 187]).

4. TESTING REACHABILITY

Given a system $\Sigma = (A, B)$, we recall from Theorem 3.2 *(i)* that Σ is reachable if and only if $\mathcal{I} = \mathcal{R}[z]$. According to the Hilbert Nullstellensatz this is equivalent to the condition that $\mathcal{V}(\mathcal{I}) = \emptyset$. To test this condition, we can use the ideals $\mathcal{H}$ and $\mathcal{J}$ instead, because these ideals have the same variety as $\mathcal{I}$. Using the methods of the previous section, we can compute a Gröbner basis of the ideals $\mathcal{J}$ and $\mathcal{H}$. The system $\Sigma = (A, B)$ is reachable if and only if these Gröbner bases contain a nonzero constant polynomial (i.e. a nonzero element of the field $\mathcal{K}$).

If one is only interested in the reachability of a system, the verification method based on the ideal $\mathcal{H}$ can be modified to obtain a more efficient algorithm.

PROPOSITION 4.1. ([7, p. 178]) *Let $\Sigma = (A, B)$ be a system over $\mathcal{R}$. Introduce an n-rowvector $q^T = (q_1 \cdots q_n)$ of new indeterminates and define*

$$(p_1 \cdots p_{n+m}) := (q_1 \cdots q_n) \cdot (zI - A \mid B).$$

Consider $p_1, \ldots, p_{n+m}$ *as polynomials in* $\mathcal{R}[z, q_1, \ldots, q_n]$. *Then*

$$
\begin{array}{ll}
 & (zI - A \mid B) \text{ is right-invertible over } \mathcal{R}[z], \\
\Longleftrightarrow & \\
 & \text{The reduced Gröbner basis of the ideal } \langle p_1, \ldots, p_{n+m} \rangle \text{ in} \\
 & \mathcal{R}[z, q_1, \ldots, q_n] \text{ is } G = \{q_1, \ldots, q_n\}, \\
 & \text{independent of the chosen term ordering.}
\end{array}
$$

□

So, to test the right-invertibility of $(zI - A \mid B)$, only *one* Gröbner basis has to be computed. Recall that in the original method based on the ideal $\mathcal{H}$, at least n Gröbner bases have to be computed, since for every ideal $\mathcal{H}_i$ $(i = 1, \ldots, n)$ a Gröbner basis has to be determined.

The reachability test described in Proposition 4.1 also has an other advantage: it can be used for the computation of a right-inverse of $(zI - A \mid B)$ over $\mathcal{R}[z]$.

PROPOSITION 4.2. *Let* $\Sigma = (A, B)$ *be a system over* $\mathcal{R}$, *and* $q^T = (q_1 \cdots q_n)$ *be an* n*-rowvector of new indeterminates. Let* $M(z, q_1, \ldots, q_n)$ *be an* $(n+m) \times n$ *matrix over* $\mathcal{R}[z, q_1, \ldots, q_n]$ *such that*

$$
(q_1 \cdots q_n) \cdot (zI - A \mid B) \cdot M(z, q_1, \ldots, q_n) = (q_1 \cdots q_n). \tag{12}
$$

Then the matrix $M_0(z) \in \mathcal{R}[z]^{(n+m) \times n}$, *obtained after substitution of* $q_1 = q_2 = \cdots = q_n = 0$ *in* $M(z, q_1, \ldots, q_n)$, *is a right-inverse of* $(zI - A \mid B)$ *over* $\mathcal{R}[z]$.

PROOF

Assume that (12) holds, and introduce a new indeterminate λ. Then (12) remains valid when $(q_1, \ldots, q_n)$ is replaced by $(\lambda q_1, \ldots, \lambda q_n)$:

$$
(\lambda q_1 \cdots \lambda q_n) \cdot (zI - A \mid B) \cdot M(z, \lambda q_1, \ldots, \lambda q_n) = (\lambda q_1 \cdots \lambda q_n).
$$

Subtracting $(\lambda q_1 \cdots \lambda q_n)$ on both left- and right-hand side, and factorizing the common term λ, we obtain

$$
\lambda \cdot (q_1 \cdots q_n) \cdot ((zI - A \mid B) \cdot M(z, \lambda q_1, \ldots, \lambda q_n) - I) = 0. \tag{13}
$$

(13) can be considered as a polynomial rowvector in the indeterminate λ. All entries of this vector are zero, so, in particular, the linear terms are zero. The coefficients of the linear terms in λ are obtained by substitution of $\lambda = 0$ in $(q_1 \cdots q_n) \cdot ((zI - A \mid B) M(z, \lambda q_1, \ldots, \lambda q_n) - I)$. This implies that $(q_1 \cdots q_n) \cdot ((zI - A \mid B) M_0(z) - I) = 0$. Since $(q_1 \cdots q_n)$ is a vector of indeterminates, we conclude that $(zI - A \mid B) M_0(z) - I = 0$.

□

If Proposition 4.1 is used to test the right-invertibility of $(zI - A \mid B)$, a reduced Gröbner basis G of the ideal $\langle p_1, \ldots, p_{n+m} \rangle$ has to be computed. Buchberger's algorithm (see [1], [2]) does not only yield a Gröbner basis of this ideal, but also the polynomial coefficients, describing the relationship between the polynomials $p_1, \ldots, p_{n+m}$, and the polynomials in the Gröbner basis G. If $G = \{q_1, \ldots, q_n\}$ this implies that also a matrix $M(z, q_1, \ldots, q_n) \in \mathcal{R}[z, q_1, \ldots, q_n]^{(n+m)\times n}$ is obtained, such that (12) holds. Then Proposition 4.2 can be applied to obtain a right-inverse of $(zI - A \mid B)$ over $\mathcal{R}[z]$. This result is very interesting from the control point of view, since a right-inverse of $(zI - A \mid B)$ is required in the constructive solution of several important control problems.

5. Testing stabilizability

According to Theorem 3.2 *(ii)*, a system $\Sigma = (A, B)$ is stabilizable with respect to the Hurwitz set $\mathcal{D}$ if and only if $\mathcal{I} \cap \mathcal{D} \neq \emptyset$. In general it is difficult to verify this condition. Although a characterization of the ideal $\mathcal{I}$ can be obtained using Gröbner bases, the Hurwitz set $\mathcal{D}$ remains a complicated object: it is only a multiplicative set. Therefore it is difficult to solve the stabilizability question in full generality.

In a lot of interesting cases the Hurwitz set $\mathcal{D}$ has a special structure, and stabilizability can be translated into a condition on the variety $\mathcal{V}(\mathcal{I})$ of the ideal $\mathcal{I}$. For example, consider a time-delay system with k incommensurable time-delays $\tau_1, \ldots, \tau_k$, as described in Example 1.2. Define the set $W \subset \mathbb{C}^{k+1}$ as

$$W = \left\{ (\lambda, e^{-\tau_1 \lambda}, \ldots, e^{-\tau_k \lambda}) \mid \lambda \in \overline{\mathbb{C}^+} \right\}. \tag{14}$$

Then the Hurwitz set $\mathcal{D}$, describing the classical notion of stability for time-delay systems of the form (2), is given by

$$\begin{aligned} \mathcal{D} \;\; = \;\; & \{p(z, s_1, \ldots, s_k) \in \mathbb{R}[z, s_1, \ldots, s_k] \mid p \text{ is monic in } z \\ & \text{and } \forall\, \alpha \in W : p(\alpha) \neq 0\}. \end{aligned} \tag{15}$$

So W is the set of points in which a stable polynomial is not allowed to have zeros.

In the next theorem we reformulate an earlier result from [5] within our algebraic framework.

Theorem 5.1. ([5]) *Let $\Sigma = (A, B)$ be a time-delay system with k incommensurable time-delays $\tau_1, \ldots, \tau_k$, modeled as a system over the ring $\mathcal{R} = \mathbb{R}[s_1, \ldots, s_k]$. Let $\mathcal{I}$ be the ideal associated with Σ as described in (6), and define W and $\mathcal{D}$ as in (14) and (15), respectively. Then Σ is stabilizable by dynamic state feedback if and only if*

$$\mathcal{V}(\mathcal{I}) \cap W = \emptyset. \tag{16}$$

□

In the original paper of Emre and Knowles ([5]), condition (16) was given in the form of a generalized Hautus test:

$$\forall \lambda \in \overline{\mathbb{C}^+} : \mathrm{rank}(\lambda I - A(e^{-\tau_1\lambda}, \ldots, e^{-\tau_k\lambda}) \mid B(e^{-\tau_1\lambda}, \ldots, e^{-\tau_k\lambda})) = n.$$

However, condition (16) is more appropriate for algorithmic verification. First, the variety $\mathcal{V}(\mathcal{I})$ can be computed using Gröbner basis techniques. Also the ideal $\mathcal{J}$ can be used for this purpose. This is advantageous from the computational point of view, because a Gröbner basis of this ideal is easier to compute in general. If $\mathcal{V}(\mathcal{I})$ is zero-dimensional, the test is rather simple: one has to verify whether a finite number of points (i.e. all points of $\mathcal{V}(\mathcal{I})$, calculated using the Gröbner basis algorithm), are elements of W. However, this stabilizability test remains troublesome for higher-dimensional varieties.

In [8], the generic dimension of the variety $\mathcal{V}(\mathcal{I})$ (in the algebraic geometric setting, using the Zariski topology) was studied. If $\mathcal{R} = \mathbb{R}[s_1, \ldots, s_k]$ and $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$, it turns out that $\mathcal{V}(\mathcal{I})$ is generically empty if $k < m$. Furthermore, if $k \geq m$, the generic dimension of $\mathcal{V}(\mathcal{I})$ is $k - m$. In particular this implies that for systems with commensurable time-delays ($k = 1$) we may expect the variety $\mathcal{V}(\mathcal{I})$ to be empty if $m \geq 2$, and zero-dimensional if $m = 1$.

Although the question of stabilizability for time-delay systems may be transformed into a condition on the variety $\mathcal{V}(\mathcal{I})$, the construction of a stabilizing compensator remains a difficult problem. For this a polynomial $p \in \mathcal{I} \cap \mathcal{D}$ is required. The condition $\mathcal{V}(\mathcal{I}) \cap W = \emptyset$ guarantees the existence of a stable polynomial in $\mathcal{I}$, but does not yield a constructive method for the computation of a polynomial $p \in \mathcal{I} \cap \mathcal{D}$.

6. Conclusion

In this paper a unified approach to test the reachability and stabilizability of systems over (polynomial) rings was proposed. The main idea is the introduction of an ideal $\mathcal{I}$, describing the set of internal dynamics obtainable from the original system by applying dynamic state feedback. The ideal $\mathcal{I}$ can be approximated by other ideals, that may be computed using Gröbner basis methods. In this way, also the variety $\mathcal{V}(\mathcal{I})$ of the ideal $\mathcal{I}$ is obtained. Based on the outcome of these calculations, the reachability and stabilizability of a system can be verified.

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